CS 4300
Computer Graphics

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CS4300

Lectures 13,14 – October 3, 4, 10 2012

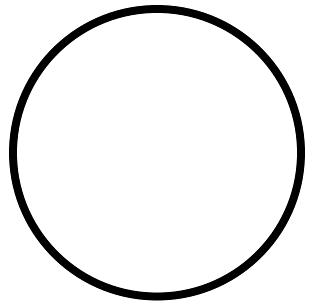
Today's Topics

- Curves
- Fitting Curves to Data Points
- Splines
- Hermite Cubics
- Bezier Cubics

Curves

A *curve* is the continuous image of an interval in *n*-space.

Implicit $f(x, y) = 0$

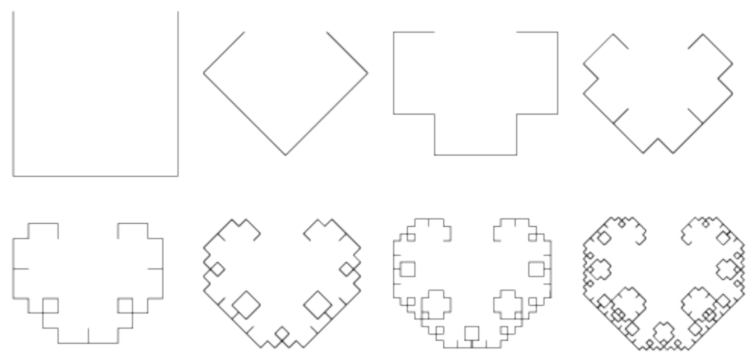


$x^2 + y^2 - R^2 = 0$

Parametric $(x(t), y(t)) = \mathbf{P}(t)$

$\mathbf{P}(t) = t\mathbf{A} + (1-t)\mathbf{B}$

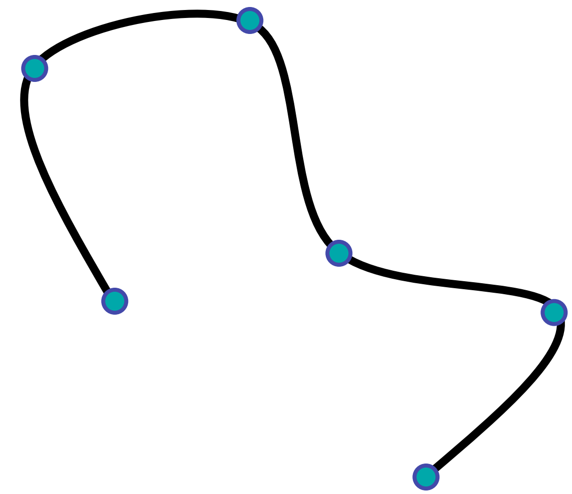
Generative $proc \rightarrow (x, y)$



Curve Fitting

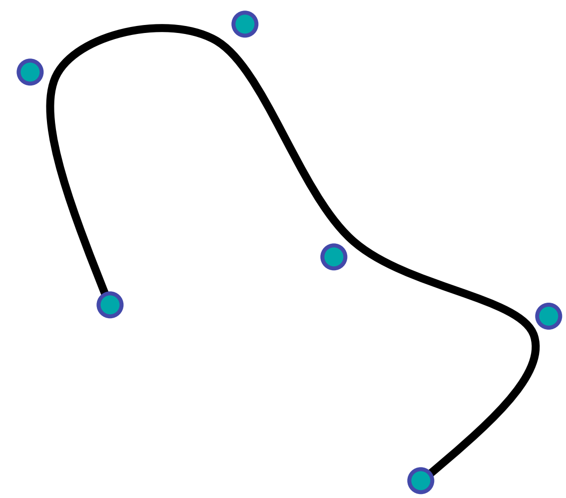
We want a curve that passes through control points.

interpolating curve



Or a curve that passes near control points.

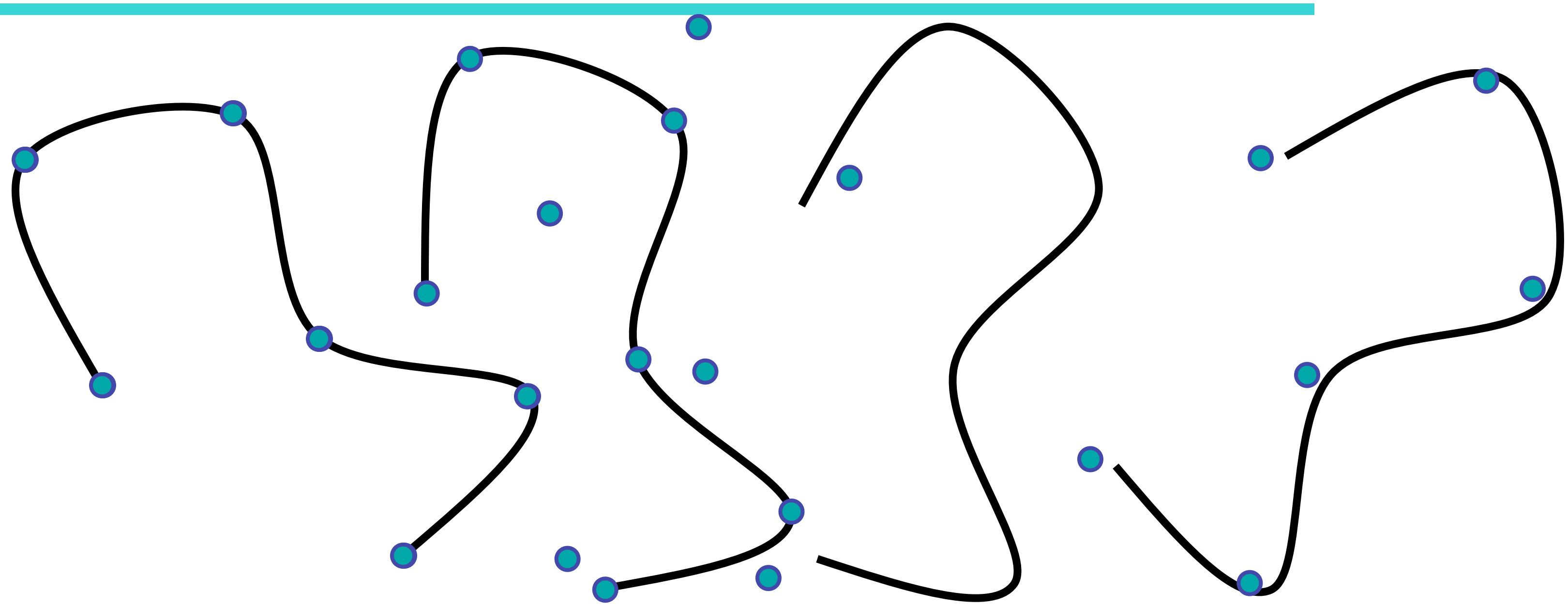
approximating curve



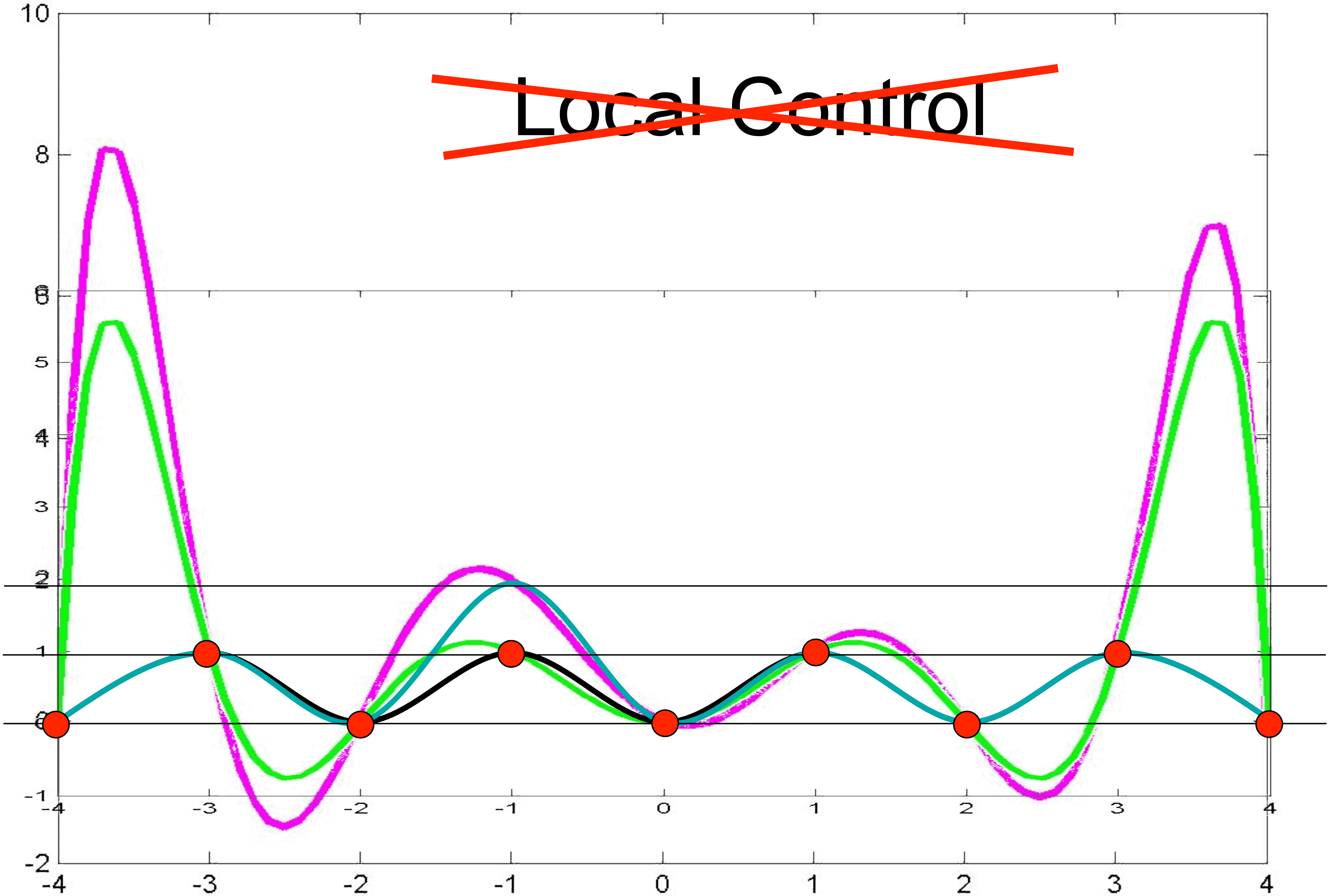
How do we create a good curve?

What makes a good curve?

Axis Independence

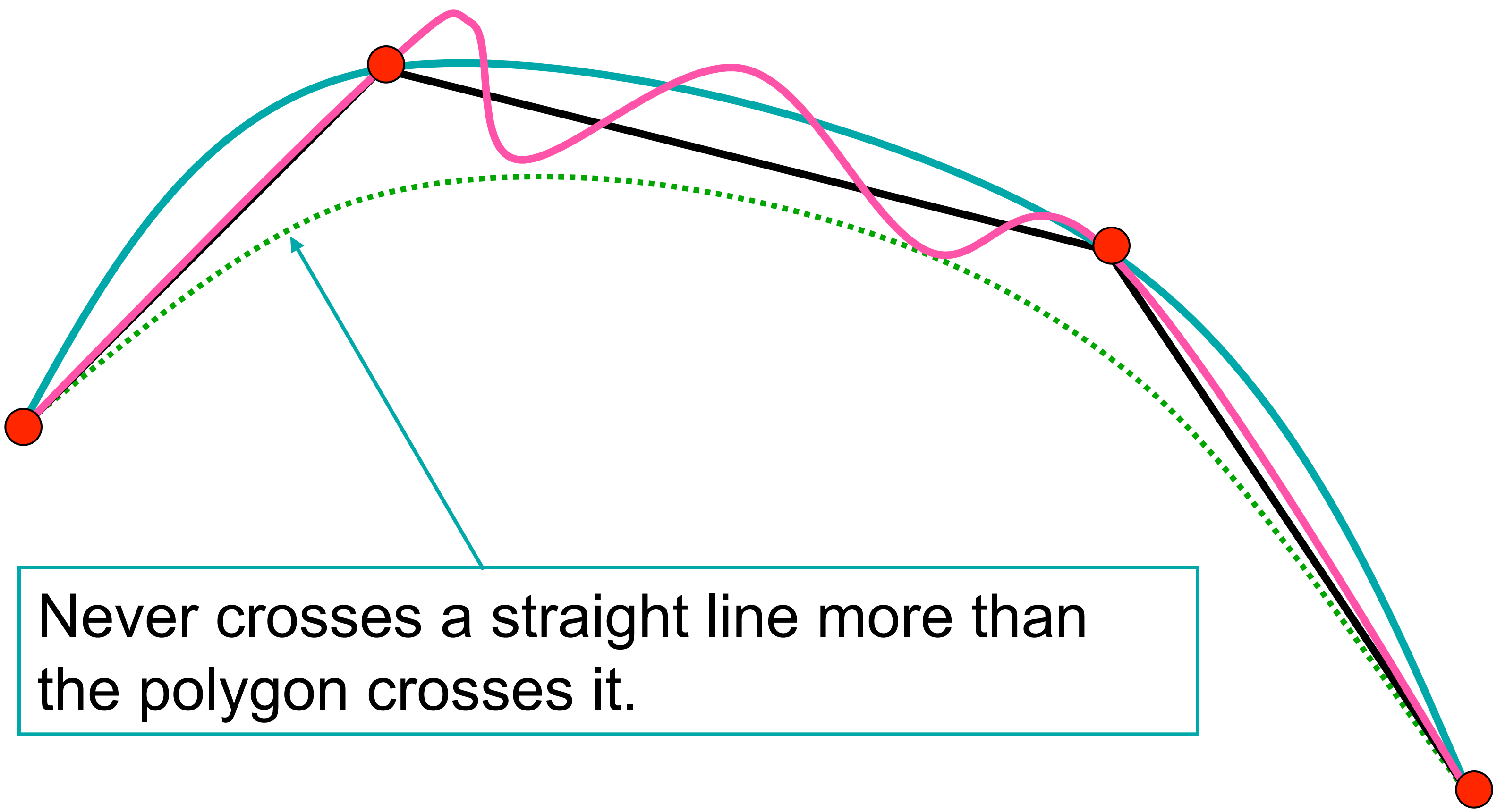


If we rotate the set of control points, we should get the rotated curve.

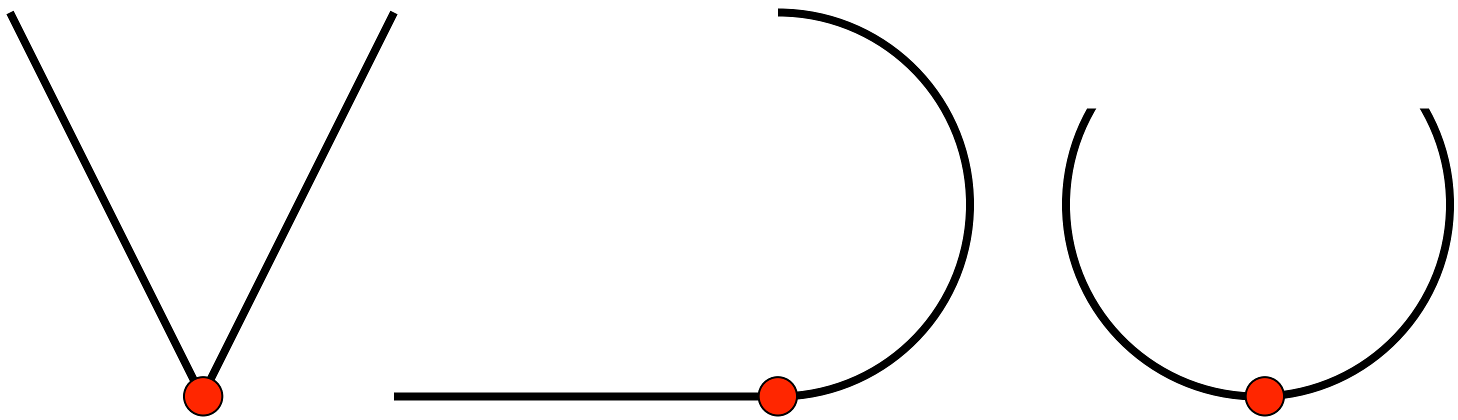


Local Control
10
8
6
4
2
0
-2
-4
-3
-2
-1
0
1
2
3
4

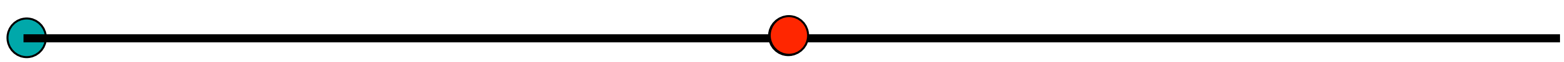
~~Variation Diminishing~~



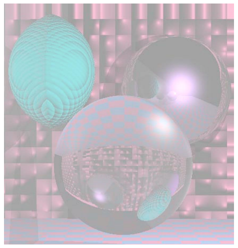
Continuity



C^0 continuity C^1 continuity C^2 continuity



G^2 continuity Not C^2 continuity



How do we Fit Curves?

The *Lagrange interpolating polynomial* is the polynomial of degree n-1 that passes through the n points,

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

and is given by

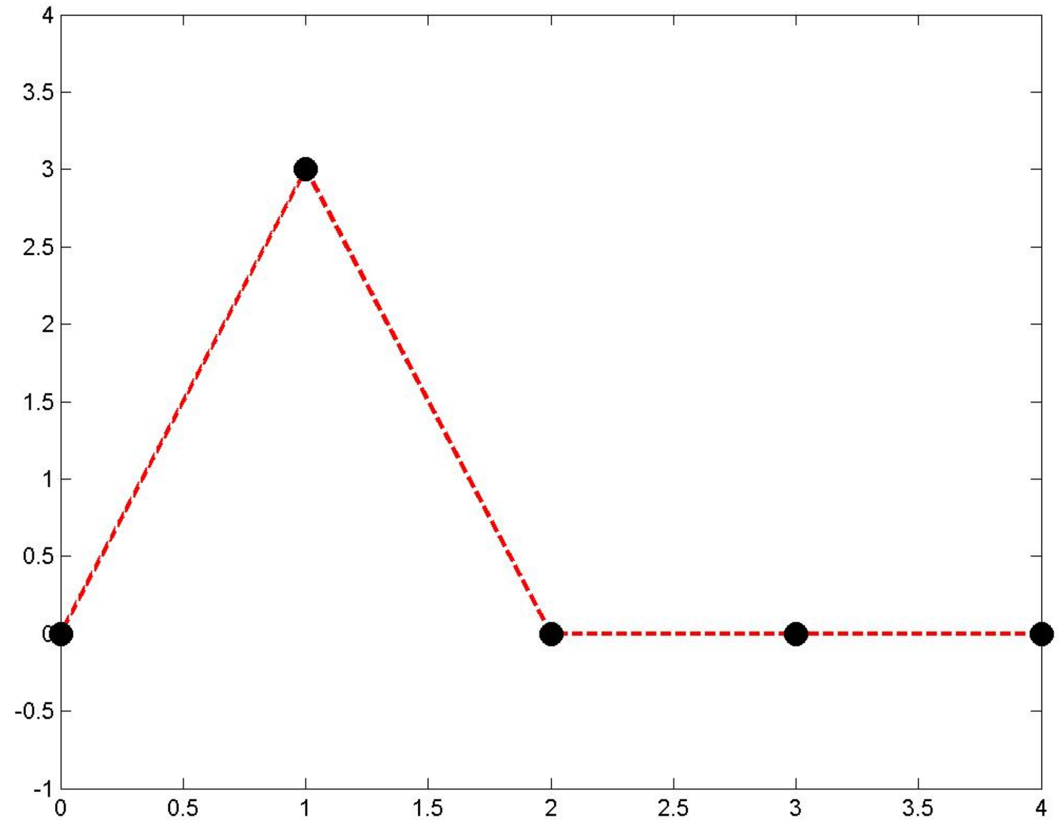
$$\mathrm{P}(\mathrm{x}) = y_1 \frac{(x-x_2)\cdots(x-x_n)}{(x_1-x_2)\cdots(x_1-x_n)} + y_2 \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} + \cdots$$

$$+ y_n \frac{(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_1)\cdots(x_n-x_{n-1})}$$

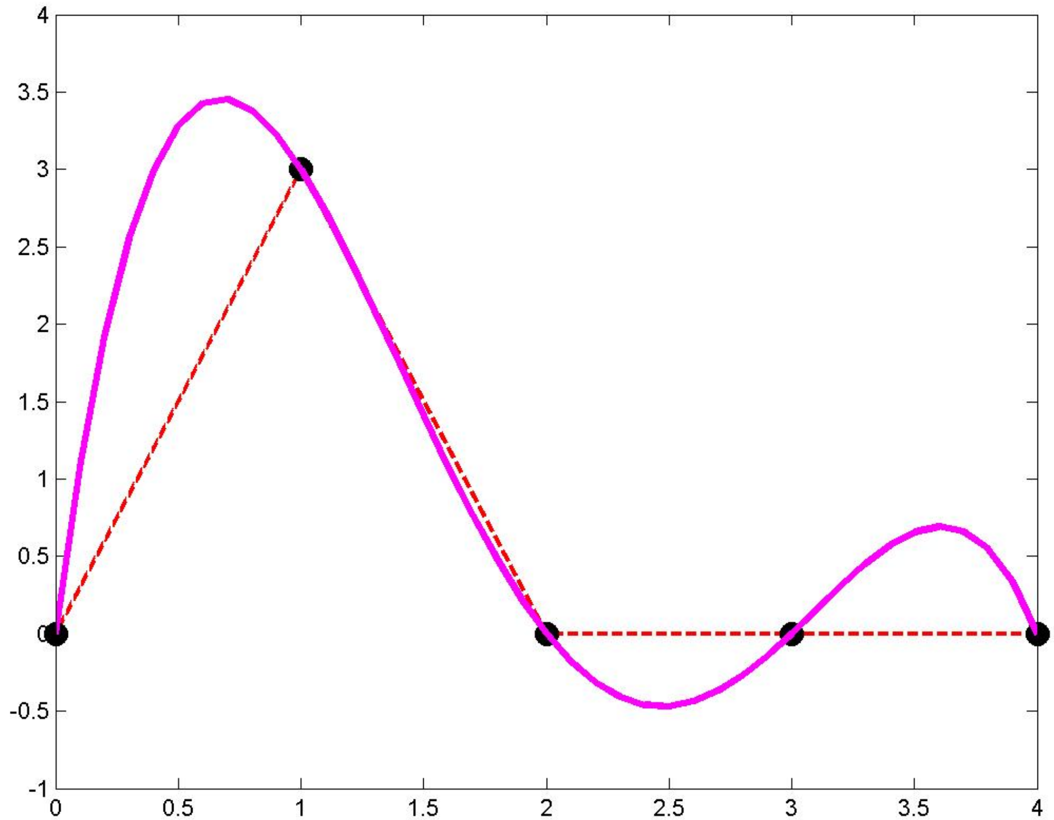
$$= \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{(x-x_j)}{(x_i-x_j)}$$

[Lagrange Interpolating Polynomial from mathworld]

Example 1

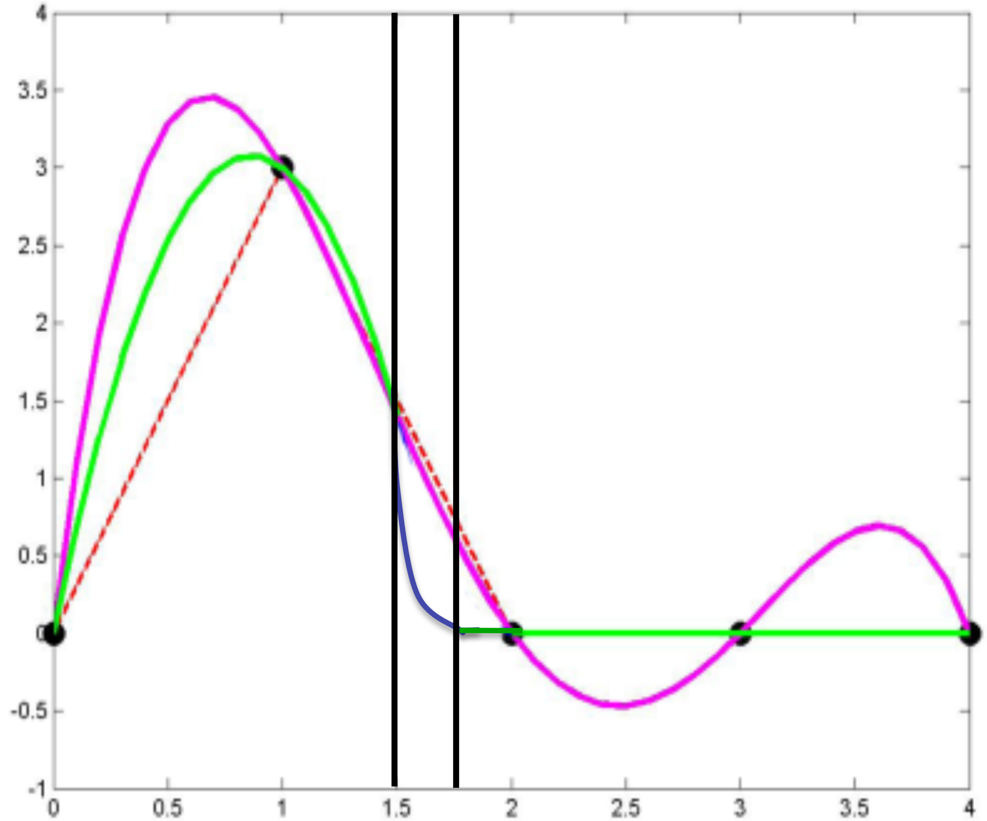


Polynomial Fit



$$P(x) = -.5x(x-2)(x-3)(x-4)$$

Piecewise Fit



$$P_a(x) = 4.1249\,x\,(x - 1.7273)$$

$$0 \le x \le 1.5$$

$$P_b(x) = 5.4\,x\,(x - 1.7273)$$

$$1.5 \le x \le 1.7273$$

$$P_c(x) = 0$$

$$1.7273 \le x \le 4$$

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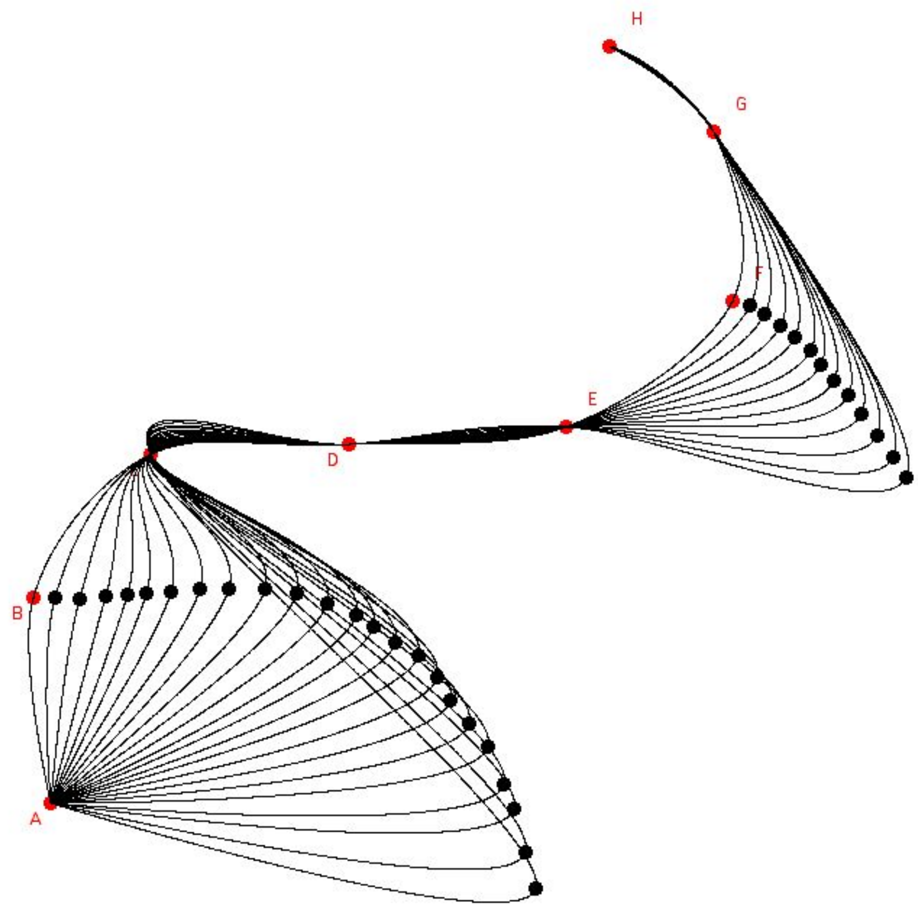
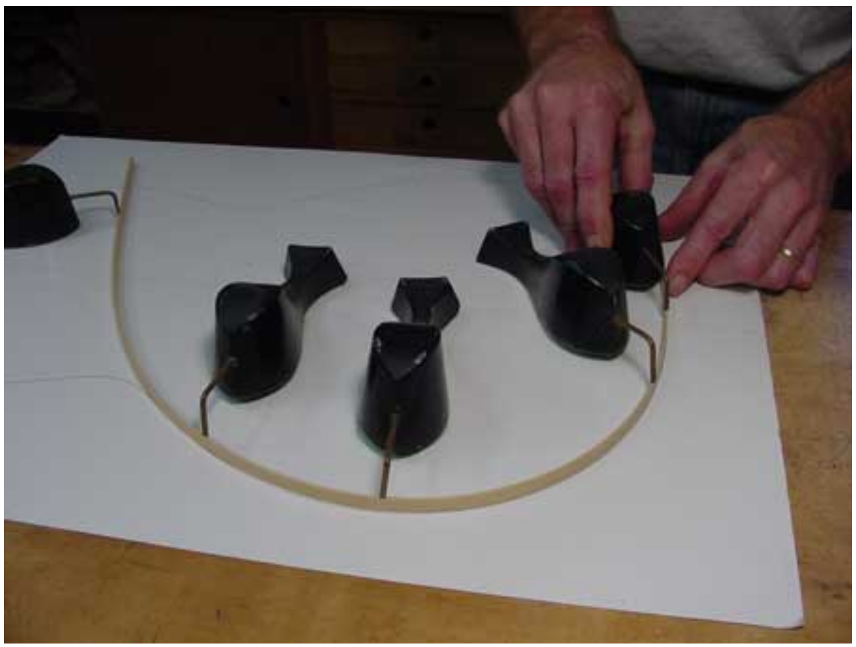
Spline Curves



Splines and Spline Ducks

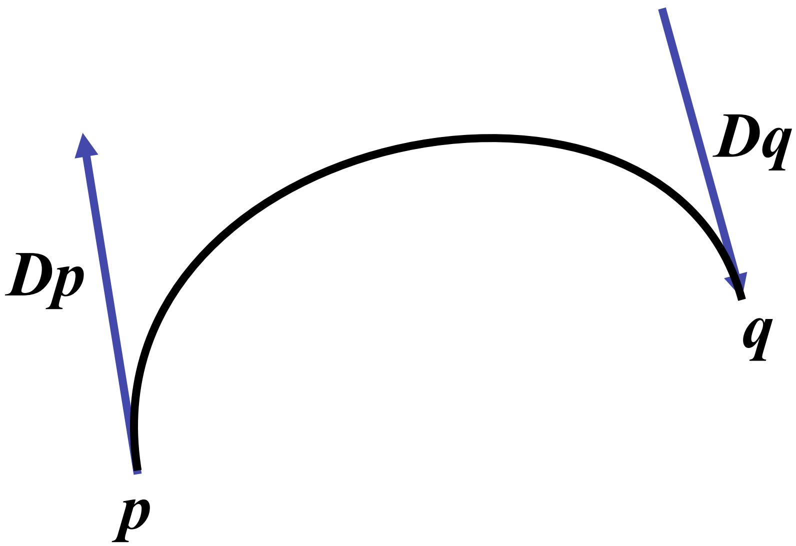


Marine Drafting Weights
http://www.frets.com/FRETSPages/Luthier/TipsTricks/DraftingWeights/draftweights.html

Drawing Spline Today (esc)

1. Draw some curves in PowerPoint.
2. Look at Perlin's B-Spline Applet.

Hermite Cubics



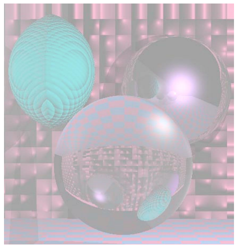
$$\boldsymbol{P}(t) = at^3 + bt^2 + ct + d$$

$$\boldsymbol{P}(0) = \boldsymbol{p}$$

$$\boldsymbol{P}(1) = \boldsymbol{q}$$

$$\boldsymbol{P}'(0) = \boldsymbol{Dp}$$

$$\boldsymbol{P}'(1) = \boldsymbol{Dq}$$



Hermite Coefficients

$\boldsymbol{P}(t) = at^3 + bt^2 + ct + d$

$\boldsymbol{P}(0) = \boldsymbol{p}$

$\boldsymbol{P}(1) = \boldsymbol{q}$

$\boldsymbol{P}'(0) = \boldsymbol{Dp}$

$\boldsymbol{P}'(1) = \boldsymbol{Dq}$

$$\boldsymbol{P}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\boldsymbol{P}'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

For each coordinate, we have 4 linear equations in 4 unknowns

Boundary Constraint Matrix

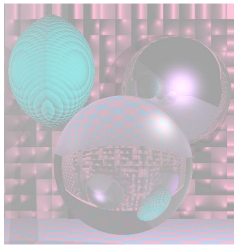
$$\boldsymbol{P}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\boldsymbol{P'}(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{Dp} \\ \boldsymbol{Dq} \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Hermite Matrix

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{M_H}} \underbrace{\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{Dp} \\ \boldsymbol{Dq} \end{bmatrix}}_{\boldsymbol{G_H}}$$



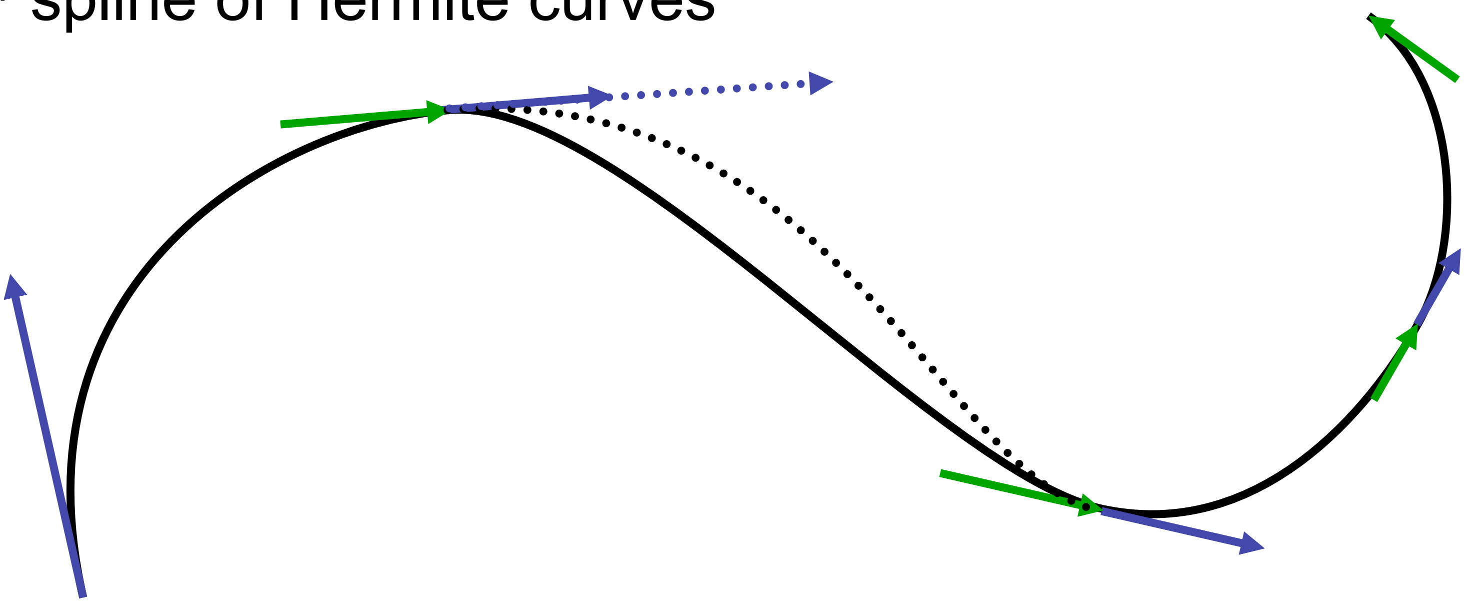
Hermite Blending Functions

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_H \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{Dp} \\ \boldsymbol{Dq} \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{Dp} \\ \boldsymbol{Dq} \end{bmatrix}$$

$$P(t) = \boldsymbol{p} \qquad + \boldsymbol{q} \qquad + \boldsymbol{Dp} \qquad + \boldsymbol{Dq}$$

Splines of Hermite Cubics

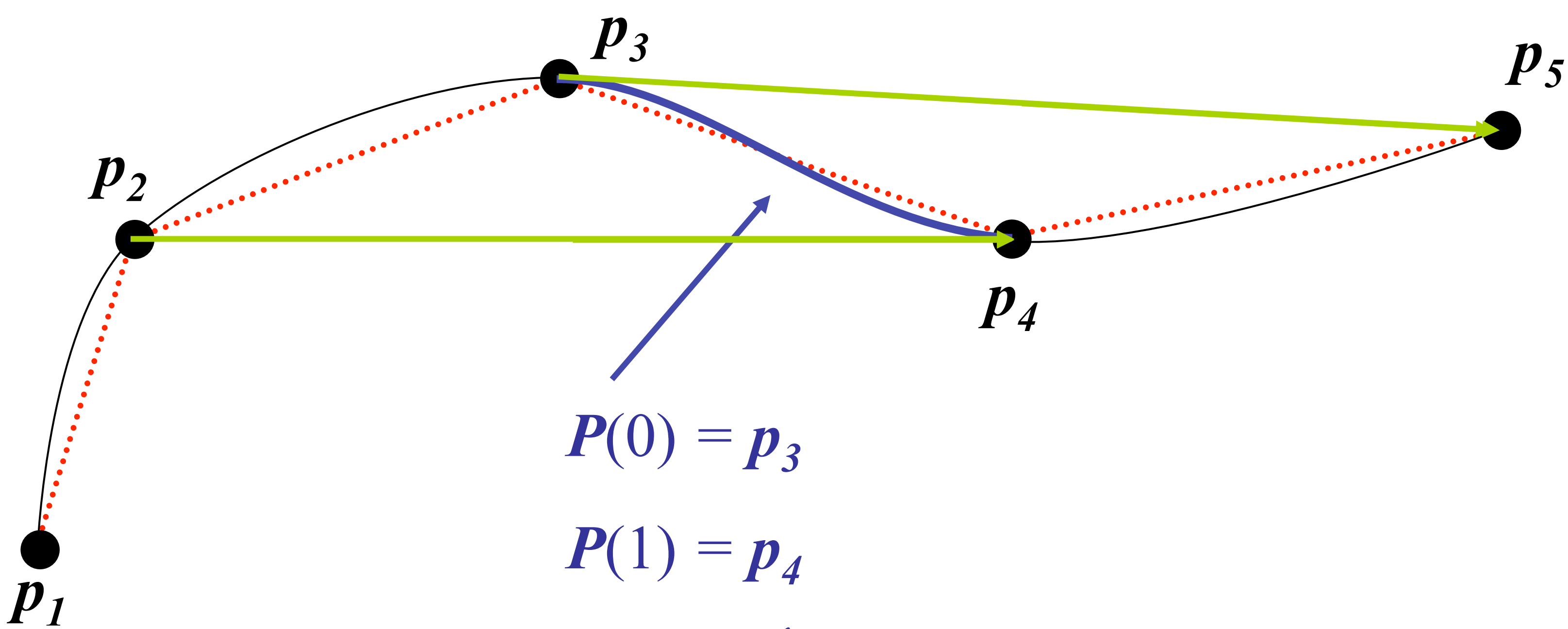
a C^1 spline of Hermite curves



a G^1 but not C^1 spline of Hermite curves

The vectors shown are 1/3 the length of the tangent vectors.

Computing the Tangent Vectors
Catmull-Rom Spline



$$P(0) = p_3$$

$$P(1) = p_4$$

$$P'(0) = \tfrac{1}{2}(p_4 - p_2)$$

$$P'(1) = \tfrac{1}{2}(p_5 - p_3)$$

Cardinal Spline

The Catmull-Rom spline

$\boldsymbol{P}(0) = \boldsymbol{p}_3$

$\boldsymbol{P}(1) = \boldsymbol{p}_4$

$\boldsymbol{P}'(0) = \frac{1}{2}(\boldsymbol{p}_4 - \boldsymbol{p}_2)$

$\boldsymbol{P}'(1) = \frac{1}{2}(\boldsymbol{p}_5 - \boldsymbol{p}_3)$

is a special case of the Cardinal spline

$\boldsymbol{P}(0) = \boldsymbol{p}_3$

$\boldsymbol{P}(1) = \boldsymbol{p}_4$

$\boldsymbol{P}'(0) = (1 - t)(\boldsymbol{p}_4 - \boldsymbol{p}_2)$

$\boldsymbol{P}'(1) = (1 - t)(\boldsymbol{p}_5 - \boldsymbol{p}_3)$

$0 \le t \le 1$ is the *tension*.

Drawing Hermite Cubics

$$\boldsymbol{P}(t) = \boldsymbol{p}\left(2t^3 - 3t^2 + 1\right) + \boldsymbol{q}\left(-2t^3 + 3t^2\right) + \boldsymbol{Dp}\left(t^3 - 2t^2 + t\right) + \boldsymbol{Dq}\left(t^3 - t^2\right)$$

- How many points should we draw?
- Will the points be evenly distributed if we use a constant increment on t ?

- We actually draw Bezier cubics.

General Bezier Curves

Given $n+1$ control points $\boldsymbol{p}_i$

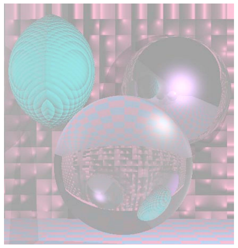
$$\boldsymbol{B}(t) = \sum_{k=0}^{n} \binom{n}{k} \boldsymbol{p_k} (1-t)^{n-k}\, t^k \qquad 0 \le t \le 1$$

where

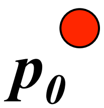
$$b_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad k = 0, \cdots n$$

$$b_{k,n}(t) = (1-t)\, b_{k,n-1}(t) + t b_{k-1,n-1}(t) \quad 0 \le k < n$$

We will only use cubic Bezier curves, $n = 3$.

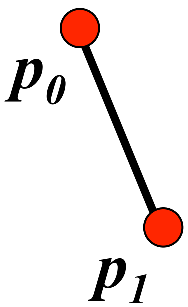


Low Order Bezier Curves



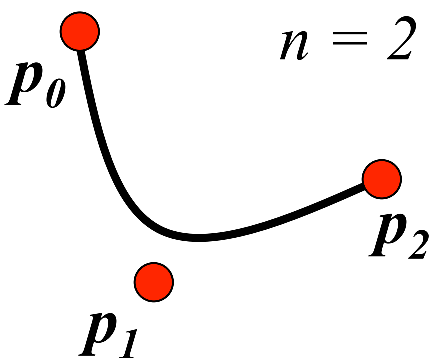
$n = 0$ $\qquad b_{0,0}(t) = 1$

$$\boldsymbol{B}(t) = \boldsymbol{p_0}\, b_{0,0}(t) = \boldsymbol{p_0} \qquad 0 \le t \le 1$$



$n = 1$ $\qquad b_{0,1}(t) = 1 - t \qquad b_{1,1}(t) = t$

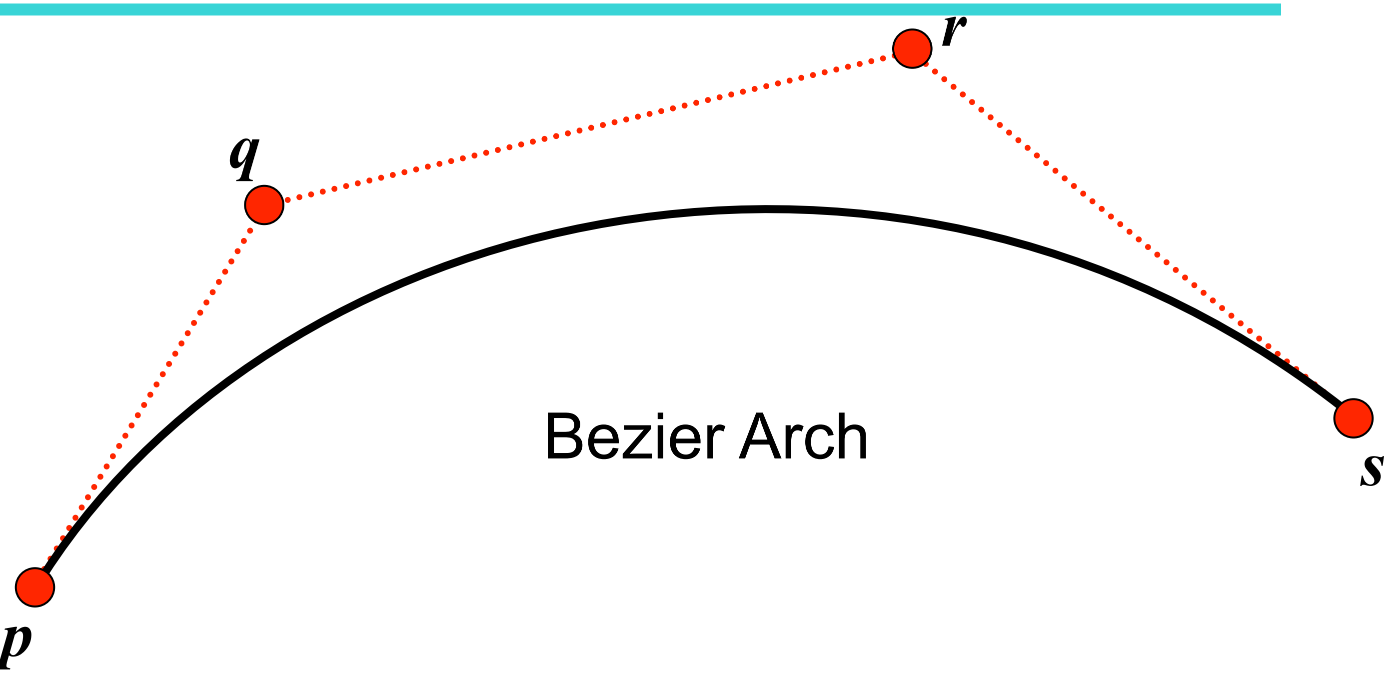
$$\boldsymbol{B}(t) = (1 - t)\, \boldsymbol{p_0} + t\, \boldsymbol{p_1} \qquad 0 \le t \le 1$$



$n = 2$ $\qquad b_{0,2}(t) = (1 - t)^2 \qquad b_{1,2}(t) = 2t\,(1 - t) \qquad b_{2,2}(t) = t^2$

$$\boldsymbol{B}(t) = (1 - t)^{2}\, \boldsymbol{p_0} + 2t\,(1 - t)\boldsymbol{p_1} + t^2\, \boldsymbol{p_2} \qquad 0 \le t \le 1$$

Bezier Curves



$n = 3$ $\qquad b_{0,3}(t) = (1 - t)^3 \qquad b_{1,3}(t) = 3t\,(1 - t)^2$

$b_{2,3}(t) = 3t^2(1 - t) \qquad b_{2,3}(t) = t^3$

$$\boldsymbol{B}(t) = (1 - t)^{3}\,\boldsymbol{p} + 3t\,(1 - t)^2\boldsymbol{q} + 3t^2(1 - t)\boldsymbol{r} + t^3\boldsymbol{s} \qquad 0 \le t \le 1$$

Bezier Matrix

$\boldsymbol{B}(t) = (1 - t)^3 \boldsymbol{p} + 3t(1 - t)^2 \boldsymbol{q} + 3t^2(1 - t)\boldsymbol{r} + t^3 \boldsymbol{s} \qquad 0 \le t \le 1$

$\boldsymbol{B}(t) = \boldsymbol{a}\,\mathrm{t}^3 + \boldsymbol{b}\mathrm{t}^2 + \boldsymbol{c}\mathrm{t} + \boldsymbol{d} \qquad 0 \le t \le 1$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{M_B}} \underbrace{\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \\ \boldsymbol{s} \end{bmatrix}}_{\boldsymbol{G_B}}$$

Geometry Vector

The Hermite Geometry Vector $G_H = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{Dp} \\ \boldsymbol{Dq} \end{bmatrix}$ $\quad H(t) = TM_H G_H$

The Bezier Geometry Vector $G_B = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \\ \boldsymbol{s} \end{bmatrix}$ $\quad B(t) = TM_B G_B$

$$T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$$

Properties of Bezier Curves

$$\boldsymbol{P}(0) = \boldsymbol{p} \qquad \boldsymbol{P}(1) = \boldsymbol{s}$$

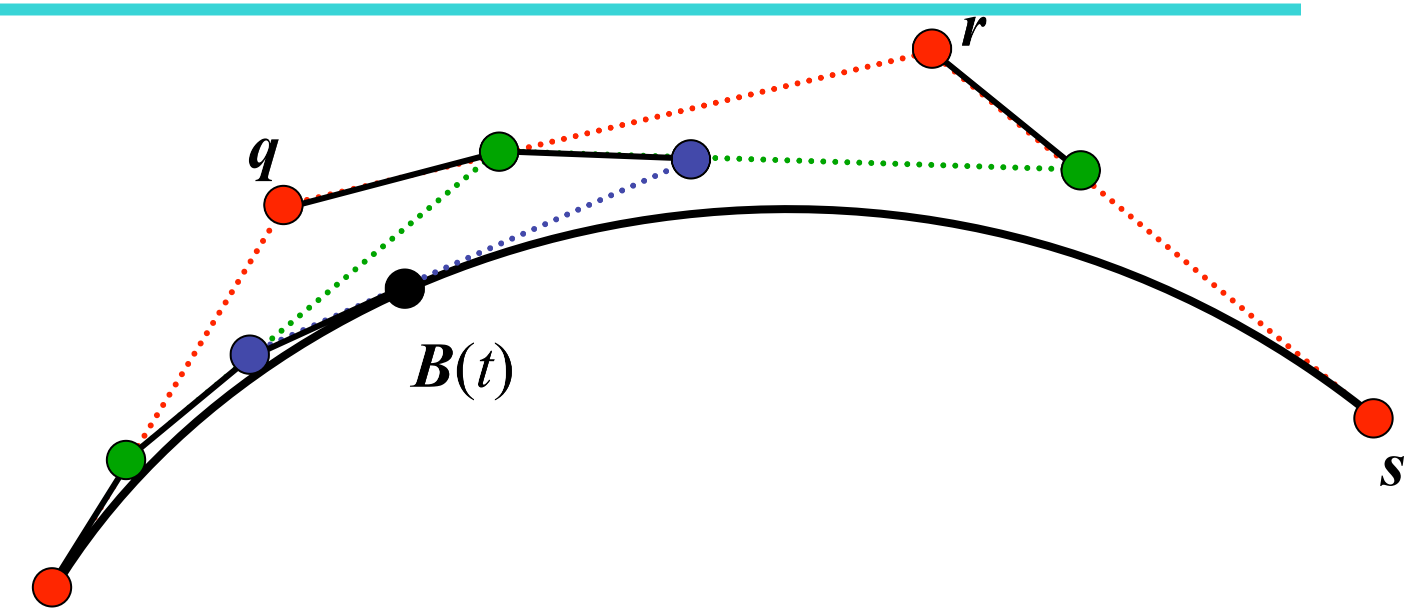
$$\boldsymbol{P}'(0) = 3(\boldsymbol{q} - \boldsymbol{p}) \qquad \boldsymbol{P}'(1) = 3(\boldsymbol{s} - \boldsymbol{r})$$

The curve is tangent to the segments ***pq*** and ***rs***.

The curve lies in the convex hull of the control points since

$$\sum_{k=1}^{3} b_{k,3}(t) = \sum_{k=1}^{3} \binom{3}{k} (1-t)^k \, t^{3-k} = \left((1-t)+t\right)^3 = 1$$

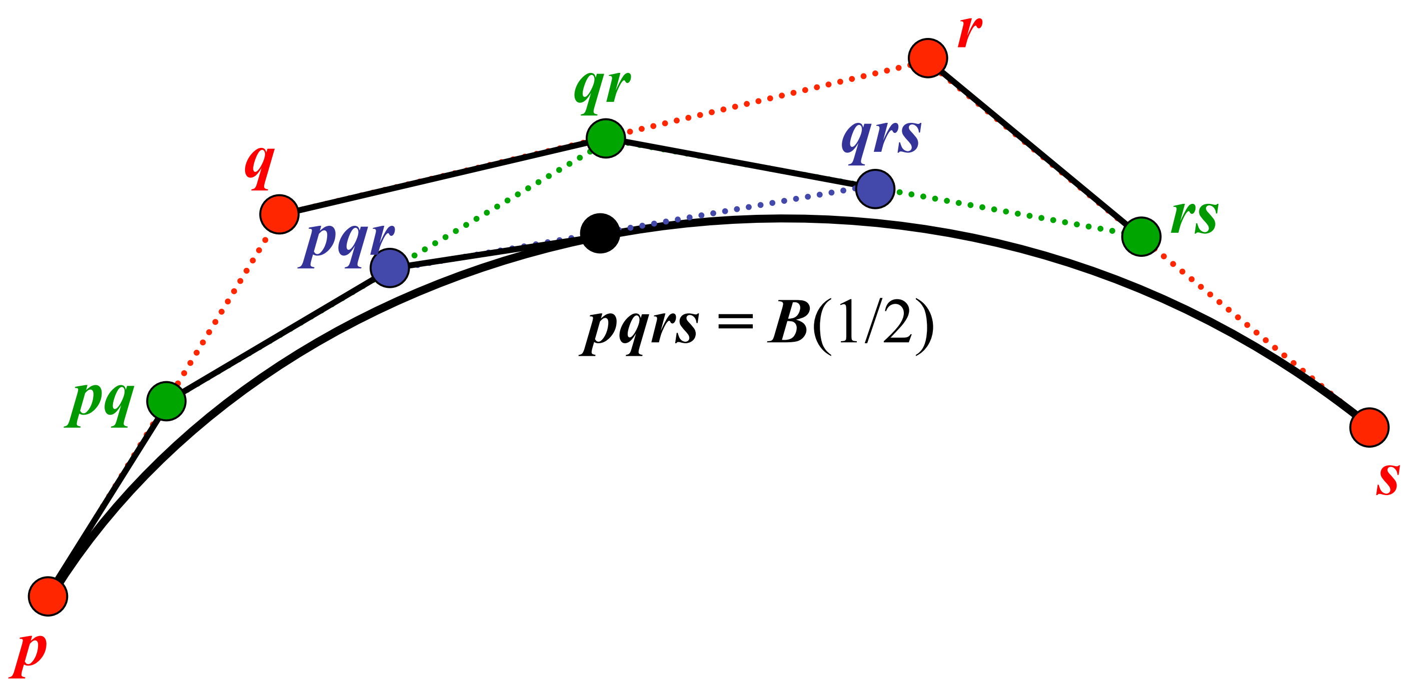
Geometry of Bezier Arches



Pick a t between 0 and 1 and go t of the way along each edge.

Join the endpoints and do it again.

Geometry of Bezier Arches



We only use $t = 1/2$.

```
drawArch(P, Q, R, S){
 if (ArchSize(P, Q, R, S) <= .5 ) Dot(P);
 else{
  PQ = (P + Q)/2;
  QR = (Q + R)/2;
  RS = (R + S)/2;

  PQR = (PQ + QR)/2;
  QRS = (QR + RS)/2;

  PQRS = (PQR + QRS)/2

  drawArch(P, PQ, PQR, PQRS);
  drawArch(PQRS, QRS, RS, S);
 }
}
```

Putting it All Together

- Bezier Arches
- Catmull-Rom Splines